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ENERGY THRESHOLDS OF NONLINEAR COUPLED OSCILLATORS

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The vibration absorber or tuned mass damper is still a very popular means to reduce excessive vibrations. Many improvements of the original concept have been reported to extend the narrow band efficiency of the device. The strongly nonlinear vibration absorber combined with a main linear oscillator is one of the intensively studied alternatives. Here, the strongly nonlinear vibration absorber is discussed as well, but in combination with a main nonlinear oscillator, a less studied option. Clearly, linearity of the main system cannot always be guaranteed at all times due to extreme environmental conditions or choices in the constructive design of the main system. An alternative system formulation is presented to analyze the nonlinear coupled oscillators. Based on the underlying Hamiltonian system, the main oscillator is replaced by a predefined motion as input to the auxiliary device. The predefined motion incorporates the nonlinear characteristic of the main system such as an amplitude dependent frequency. A parameter threshold is defined which combines main and auxiliary parameters. A bifurcation analysis reveals appropriate tuning rules. The developed method offers insight and simplifies tuning but has its limitations as the chosen predefined motion of the main system does not hold for all operating conditions. A comparison with recently reported tuning procedures is made.

1. Introduction

Vibration control of engineering structures is of paramount importance to limit vibrations to tolerable levels. Vibrations in civil structures cause disturbance, discomfort, damage and destruction. Passive vibration control is a widely accepted strategy. Passive elements do not need external power to operate, which renders the element's robustness and simplicity. Within this research, the structure to be controlled will be addressed as the main system, the passive control element as the auxiliary system. The research will be restricted to a two degree of freedom compound system.The main system is often modeled as a linear oscillator, characterized by its natural frequency. In case the auxiliary system is linear as well, a very efficient control over a narrow frequency band is obtained [1]. A frequency band in the neighborhood of the main system's natural frequency. The auxiliary system's natural frequency is chosen close to the natural frequency of the main system. An idea introduced by [2] and extended in many ways [3],[4] and references therein. A nonlinear auxiliary system with a nonlinear restoring force (spring force) is characterized by an amplitude dependent natural frequency, typically revealed by means of a harmonic balancing technique [5]. The free vibration frequency of the nonlinear auxiliary system changes as the amplitude of vibration changes. This inspired researchers to use the element for passive control such that a broader operational frequency band can be obtained. The mechanism where the auxiliary system vibrates heavily to reduce the main system's vibration amplitude is quite complex. A remarkable property is the existence of a so called energy threshold. It is a required level of initial energy of the main system above which the auxiliary system works efficient. Literature offers semi-analytical techniques to understand and quantify this behavior and uses the results for tuning purposes [6], [7],[8],[9], [5]. It can be stated that this behavior is well understood. Many dynamical systems encountered in civil engineering applications are however inherently nonlinear: nonlinear suspension bridges, inelastic behavior in case of large deformations, nonlinear behavior due to the presence of deformable mountings for vibration isolation [10],[11],[12]. If the main system is considered nonlinear as well then a set of nonlinear coupled oscillators is obtained, a less studied, yet important topic. Recent literature reports a few tuning procedures for specific choices of nonlinearities and specific choices of external forces such as impulsive and harmonic forces [16], [13]. Here a tuning procedure is presented based on a reduction of the original model to a base excited single degree of freedom model. The reduction step allows to replace the energy threshold by a parameter threshold such that a single bifurcation analysis is able to study the influence of each parameter separately. The outline of the paper is as follows: Section 2 introduces the coupled nonlinear oscillators. In Section 3 the reduction of the original model to an equivalent single degree of freedom model is explained. Section 4 contains the bifurcation analysis. In Section 5 the tuning procedure is explained. The harmonically excited coupled oscillators are discussed in Section 6. Section 7 reports a comparison with other methods. Finally, some concluding remarks are formulated in Section 8.

2. Coupled nonlinear oscillators

Figure 1 shows the coupled nonlinear oscillators. The equations of motion can be written as :

$$m\ddot{x} + kx^3 + c_{na}(\dot{x} - \dot{x}_{na}) + k_{na}(x - x_{na})^3 = 0 \tag{1}$$
$$m_{na}\ddot{x}_{na} + c_{na}(\dot{x}_{na} - \dot{x}) + k_{na}(x_{na} - x)^3 = 0 \tag{2}$$

All symbols are explained on Fig.1. The nonlinear springs are cubic springs for both oscillators. We

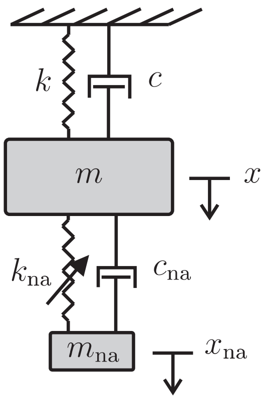

Figure 1: Nonlinear absorber attached to a nonlinear SDOF system.

study the underlying Hamiltonian system, i.e. damping is omitted. Dividing Eqs. (1)-(2) with m yields :

$$\ddot{x} + \frac{k}{m}x^3 + \epsilon\frac{k_{na}}{m_{na}}(x - x_{na})^3 = 0 \tag{3}$$
$$\ddot{x}_{na} + \frac{k_{na}}{m_{na}}(x_{na} - x)^3 = 0 \tag{4}$$

where $\epsilon = \frac{m}{m_{na}}$. First we focus on the main system without the absorber :

$$\ddot{x} + \frac{k}{m}x^3 = 0 \tag{5}$$

Classic harmonic balancing yields the free response with initial states $x(0)$ and $\dot{x}(0)$ expressed as :

$$x(t) \approx \sqrt{x(0)^2 + \frac{\dot{x}(0)^2}{\omega_{eq}^2}} \sin(\omega_{eq} t + \phi); \tag{6}$$

$$\omega_{eq}^4 = \frac{3k}{4m}\left(\omega_{eq}^2 x(0)^2 + \dot{x}(0)^2\right) \tag{7}$$

In Eq. (7) the well known amplitude dependent frequency characteristic of a cubic spring is shown. Second, we reconsider Eq. (3) without the nonlinear term $\epsilon\frac{k_{na}}{m_{na}}(x - x_{na})^3$ and replace x in the second equation with Eq. (7). Consequently, we obtain a nonlinear absorber subjected to a harmonic moving ground. It should be noted that the steps taken are restricted to sufficiently small values for ϵ. Simulations reveal a restriction to values smaller than 0.1. Our next step is an application of semi-analytical tools to understand the behavior of the base-excited nonlinear absorber.

3. Base-excited nonlinear absorber

The energy threshold, present if a cubic nonlinear spring is involved, has been extensively studied in [15]. In [8], a semi analytical approach is reported to explain the existence of the threshold. This will be applied to an alternative system formulation. Starting from the base excited nonlinear absorber :

$$\ddot{x}_{na} + \frac{k_{na}}{m_{na}}(x_{na} - x)^3 = 0 \quad ; x(t) \approx \sqrt{x(0)^2 + \frac{\dot{x}(0)^2}{\omega_{eq}^2}} \sin(\omega_{eq} t + \phi)$$

we define the relative displacement $x_r = x_{na} - x$ and obtain :

$$\ddot{x}_r + \frac{4}{3}\alpha x_r^3 \triangleq X \sin(\omega_0 t + \phi) \tag{8}$$

with

$$X = \sqrt{\omega_0^4 x(0)^2 + \omega_0^2 \dot{x}(0)^2} \tag{9}$$

$$\phi = \arctan\left(\frac{\omega_0 x(0)}{\dot{x}(0)}\right) + k\pi; \quad k \in Z \tag{10}$$

and initial conditions $x_r(0) = x_{na}(0) - x(0), \dot{x}_r(0) = \dot{x}_{na}(0) - \dot{x}(0)$. The result is a harmonically excited single degree of freedom nonlinear system with cubic spring. We introduce dimensionless time and state variables :

$$\tau = \omega_0 t$$

$$x_d = \frac{x_r \omega_0^2}{X}$$

which yields :

$$\frac{d^2 x_d}{d\tau^2} + \gamma x_d^3 = \sin(\tau + \phi) \tag{11}$$

A single parameter γ appears which combines all parameters :

$$\gamma = \tfrac{4}{3}\alpha\tfrac{X^2}{\omega_0^6} \tag{12}$$

$$= \tfrac{4}{3}\alpha\tfrac{(x(0)^2\omega_0^2 + \dot{x}(0)^2)}{\omega_0^4}. \tag{13}$$

The parameter γ will serve as bifurcation parameter to distinguish weak and strong absorber movement.

4. Bifurcation analysis

The bifurcation analysis of Eq. (11) has been reported in [5]. We briefly repeat the different steps and their result. First, complexification averaging is applied for a 1:1 resonance.

$$\rho(\tau)e^{i\tau} = \frac{dx_d(\tau)}{d\tau} + ix_d(\tau) \tag{14}$$

$\rho(\tau)$ represents a slowly varying amplitude compared to the fast time scale τ. The equation (11) is written as :

$$\begin{aligned} &\frac{d\rho}{d\tau}e^{i\tau} + \frac{i}{2}\rho e^{i\tau} - \frac{i}{2}\rho^* e^{-i\tau} + \\ &\frac{i\gamma}{8}\left[\rho^3 e^{i3\tau} - 3\rho^2\rho^* e^{i\tau} + 3\rho\rho^{*2} e^{-i\tau} - \rho^{*3} e^{-3i\tau}\right] \\ &= \frac{e^{i(\tau+\phi)} - e^{-i(\tau+\phi)}}{2i} \end{aligned} \tag{15}$$

We average over τ and introduce $\rho = \rho_{re} + i\rho_{im}$ such that real and imaginary parts are splitted :

$$\frac{d\rho_{re}}{d\tau} = \frac{\rho_{im}}{2} - \frac{3\gamma}{8}({\rho_{re}}^2 + {\rho_{im}}^2)\rho_{im} + \frac{\sin\phi}{2} \tag{16}$$

$$\frac{d\rho_{im}}{d\tau} = -\frac{\rho_{re}}{2} + \frac{3\gamma}{8}({\rho_{re}}^2 + {\rho_{im}}^2)\rho_{re} - \frac{\cos\phi}{2} \tag{17}$$

Without loss of generality, ϕ can be chosen zero due to the rotational symmetry of Eqs. (16)-(17). The next step is an analysis of the stability of the fixed points of Eqs. (16)-(17) and the closed orbits which are present for non zero initial conditions of the main system. It can be shown that the threshold from weak to strong absorber movement is at $\gamma = 0.18$. This value of γ is the transition of small orbits to orbits with large amplitude due to the presence of an unstable manifold of a homoclinic orbit. This forces the orbit to jump to high amplitude values. Details can be found in [5]. We use the existence of the threshold $\gamma = 0.18$ to tune the nonlinear absorber.

5. Tuning procedure

With γ

$$\gamma = \frac{k_{na}}{m_{na}} \frac{(x(0)^2\omega_{eq}^2 + \dot{x}(0)^2)}{\omega_{eq}^4}. \tag{18}$$

and substituting ω_{eq}^4 of Eq. (7) into Eq. (18) we find

$$\gamma = \frac{4}{3}\frac{\left(\frac{k_{na}}{m_{na}}\right)}{\left(\frac{k}{m}\right)}. \tag{19}$$

Proper absorber performance is obtained if $\gamma > 0.18$ or

$$\frac{k_{na}}{m_{na}} \geq 0.135\frac{k}{m} \tag{20}$$

A nice property is that Eq. (20) is not dependent of initial conditions. As in a Den Hartog tuning for linear absorbers, ratio's between main and auxiliary frequency are defined. Fig. 2 shows an example for the numerical values $m = 1, c = 0, k = 1, m_{na} = 0.05, c_{na} = 0$, $\dot{x}(0) = 0.5$, $x(0) = \dot{x}_{na}(0) = x_{na}(0) = 0$. The behavior is simulated for two different nonlinear spring values : $k_{na} = 0.0065$ and $k_{na} = 0.0068$. Clearly the difference of weak and strong performance can be seen which confirms the outcome of the bifurcation analysis.

We extend our experience to a harmonically excited main system.

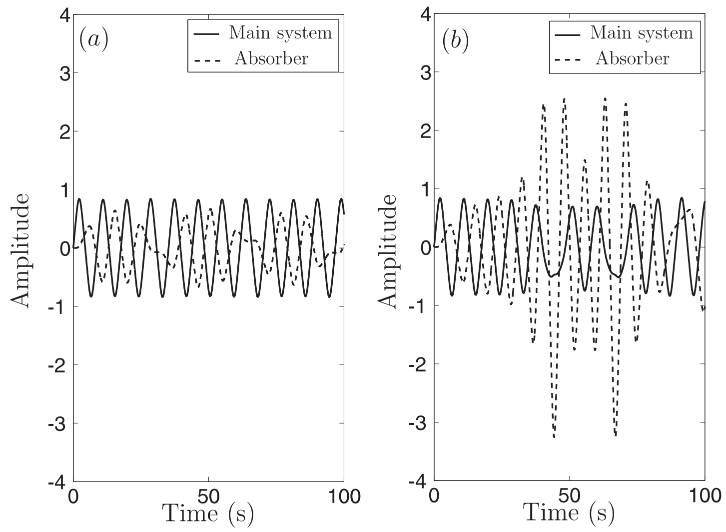

Figure 2: Response of cubic absorber attached to cubic main system ($m = 1, c = 0, k = 1, m_{na} = 0.05, c_{na} = 0$) for initial conditions $\dot{x}(0) = 0.5, x(0) = \dot{x}_{na}(0) = x_{na}(0) = 0$; (a) $k_{na} = 0.0065$ (b) $k_{na} = 0.0068$.

6. Harmonically excited system

The equations of motion for a harmonically excited main system read :

$$m\ddot{x} + c\dot{x} + kx + c_{na}(\dot{x} - \dot{x}_{na}) + k_{na}(x - x_{na})^3 = f(t) \tag{21}$$

$$m_{na}\ddot{x}_{na} + c_{na}(\dot{x}_{na} - \dot{x}) + k_{na}(x_{na} - x)^3 = 0 \tag{22}$$

with $f(t) = F\sin(\omega t)$. We include damping in the model to be able to consider resonance conditions. We repeat the procedure of section 3 but replace the main system's movement as :

$$x(t) \approx a(\omega)F\sin(\omega t + \phi(\omega)) \tag{23}$$

The expression of Eq. (23) is the result of a harmonic balancing procedure for a forced single degree of freedom system with cubic nonlinear spring [14]. The alternative system formulation reads :

$$\frac{d^2x_d}{d\tau^2} + \gamma x_d^3 = \sin(\tau + \phi(\omega)) \tag{24}$$

where γ is determined by substituting $X = \omega^2 a(\omega)F$ into Eq. (12)

$$\gamma = \frac{k_{na}}{m_{na}}\frac{a^2(\omega)F^2}{\omega^2}. \tag{25}$$

The threshold $\gamma = 0.18$ can be used here as well for tuning purposes. The expression of γ contains the force amplitude as a parameter. If force amplitude and frequency are specified, the corresponding nonlinear spring can be calculated. Harmonically excited coupled nonlinear absorbers have been recently reported with respect to their tuning and performance [16], [17] and [18]. We relate those contributions to our approach.

7. Discussion

[18] compares the efficiency of the nonlinear absorber with a two degree of freedom linear absorber to mitigate the response of a nonlinear main system. The tuning procedure minimizes an error

function which is the difference between a desired frequency response and optimized frequency response over a certain bandwidth. The optimization procedure needs an initial guess for the nonlinear spring constant of the absorber. This can be offered by Eq. (25). [17] focusses on transmitted power in coupled nonlinear oscillators. They found a better vibration mitigation performance if a softening nonlinear main system is combined with a hardening absorber or vice versa. This configuration can be handled with the approach here as the main system is replaced by a response stemming from a harmonic balancing procedure. The harmonic balancing procedure can be executed with a soft spring as well. [16] considers a harmonically forced duffing main system with a nonlinear absorber. A nice and efficient tuning procedure is presented. The equal peak method of Den Hartog for linear coupled oscillators is extended to nonlinear coupled oscillators. Besides an optimization procedure, fundamental insights are shown with respect to the behavior. The technique in [16] offers optimal parameters, independent of the force amplitude and shows the effect of the force amplitude. The method shown here can offer a first indication for the nonlinear spring but does not offer an optimal solution in the sense of [16]

8. Conclusion

Two coupled nonlinear oscillators were modeled. One nonlinear oscillator represents the main system to be controlled and the other nonlinear oscillator the controlling device. An alternative system formulation for the equations of motion was introduced which allows to perform a bifurcation analysis based on a single parameter, referred to as the parameter threshold. The reduction of the equation of motion to a single degree of freedom system is obtained by replacing the main system with a specified movement. The tuning procedure obtained is attractive due to its simplicity and the ability to analyse different situations such as impulsive load, harmonic load, hard or soft springs. However the applicability is limited as the response of the main system does not correspond to the specified motion in all operating conditions or parameter choices, especially the mass ratio.

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